Why the Bridge Method Works(?)

Algebra and Calculus

In recitation (section 8, the 9:30-10:45 slot) we were talking about the bridge method. Consider factoring the example:

$$3x^2 + 7x + 4$$

The way the method works is:

1. For a polynomial $ax^2 + bx + c$, write the alternate polynomial $x^2 + bx + ac$: So we have $x^2 + 7x + 12$. We can factor this.
2. Factor the new polynomial: $x^2 + 7x + 12 = (x + 4)(x + 3)$.
3. Now substitute the old coefficient a in front of the x terms: $(3x + 4)(3x + 3)$.
4. Factor out the coefficient where convenient, then drop it: $(3x + 4)(3)(x + 1) \to (3x + 4)(x + 1)$.

So our final answer is $(3x + 4)(x + 1) = 3x^2 + 3x + 4x + 4 = 3x^2 + 7x + 4$. I was thinking about it and now I believe I understand why it works.

Remember the discriminant for a quadratic equation: $b^2 - 4ac$. Note that when comparing the polynomials $ax^2 + bx + c$ and $x^2 + bx + ac$, the discriminant does not change at all, because

$$b^2 - 4(a)(c) = b^2 - 4(1)(ac)$$

Note that between them b also does not change. Thus, we have a way to compare the solutions of the equation $ax^2 + bx + c = 0$ and $x^2 + bx + ac = 0$. Call the solutions of the first equation x_a (for the coefficient a on x^2) and the solutions of the second equation x_1 (for the coefficient of 1):

$$x_a = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{x_1}{a}$$

So the only thing different about the solutions to the two equations is the $2a$ term in the denominator for the solution to x_a (for the solution to x_1 it would just be 2 in the denominator). So now factor the polynomial for x_1. Say it's a quadratic with two roots -k and -l:

$$x_1^2 + bx_1 + ac = (x_1 + k)(x_1 + l) = 0$$

From the above know we must have

$$b = k + l$$
$$ac = kl \implies c = \frac{kl}{a}$$

Now substitute, letting $x_1 = ax_a$ (which we found earlier):

$$(x_1 + k)(x_1 + l) = (ax_a + k)(ax_a + l) = 0$$

This is the step represented by 3 above. Let's expand this:

$$a^2x_a^2 + a(k+l)x_a + kl = 0$$

Now look what happens when we factor out a:

$$a\left[ax_a^2 + (k+l)x_a + \frac{kl}{a}\right] = 0$$

Using results we found earlier,

$$a\left[ax_a^2 + bx_a + c\right] = 0$$

Thus, we have determined that factoring the new polynomial is the same as factoring the old polynomial, except that there is an additional factor of a. That's why we follow step 4, which drops this factor of a and gives us the original factored polynomial.

Try for yourself! One nice example is $5x^2 + 7x + 2$.