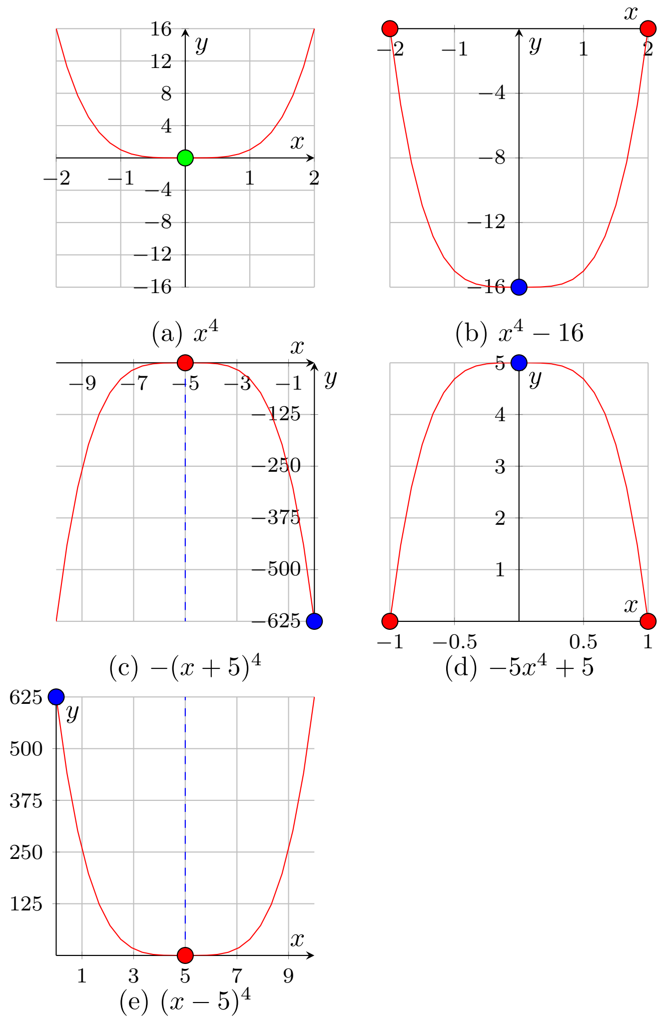
Algebra and Calculus: Homework 7 Solutions

Section 3.2: 6,22,26,32

- *Q6*: Use transformations of monomials. Transform the graph of an appropriate function of the form $y = x^n$. Indicate all x and y-intercepts.

  The x-intercept is marked with a red marker, while the y-intercept is marked with a blue marker. If they coincide, the marker is green.



(a) x^4 (b) $x^4 - 16$ (c) $-(x+5)^4$ (d) $-5x^4 + 5$ (e) $(x-5)^4$

- *Q22*: Sketch the graph of the polynomial function. Make sure the graph shows all intercepts and exhibits the proper end behavior.

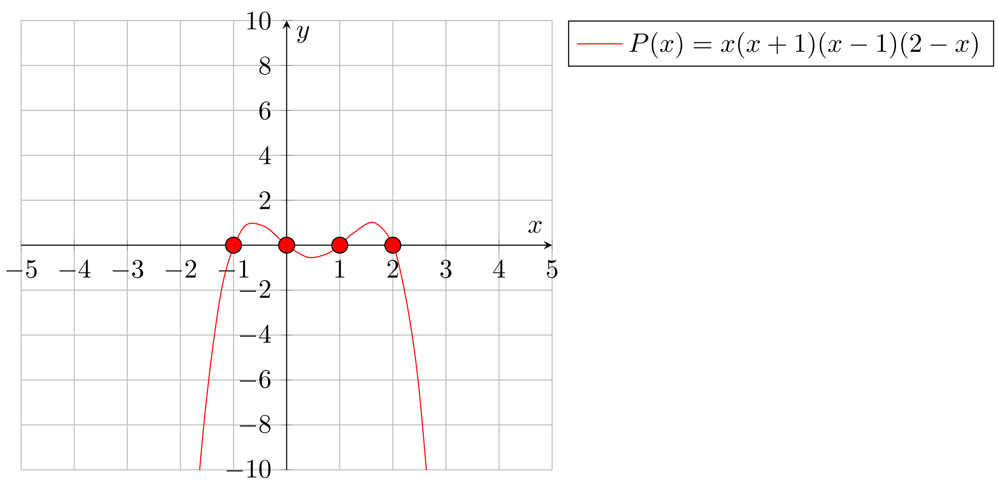
$$\begin{aligned} P(x) &= x(x+1)(x-1)(2-x) \\ &= -x(x+1)(x-1)(x-2) \\ \lim_{x\to\pm\infty} P(x) &= \lim_{x\to\pm\infty} -x^4 \end{aligned}$$

The last statement means that the end behavior of the polynomial is determined by the term of highest degree. We then expect $P(x) \to -\infty$ as $x \to \infty$ and $P(x) \to -\infty$ as $x \to -\infty$.

Because the multiplicity of each root is 1, the value of the polynomial will change sign as it crosses each root. We then expect:

$$\begin{aligned} x < -1 &: \; P(x) < 0 \\ -1 < x < 0 &: \; P(x) > 0 \\ 0 < x < 1 &: \; P(x) < 0 \\ 1 < x < 2 &: \; P(x) > 0 \\ x > 2 &: \; P(x) < 0 \end{aligned}$$

The graph below confirms this behavior:



- *Q26*: Sketch the graph of the polynomial function. Make sure the graph shows all intercepts and exhibits the proper end behavior.

$$\begin{aligned} P(x) &= -(x+1)^2(x-1)^3(x-2) \\ \lim_{x\to\pm\infty} P(x) &= \lim_{x\to\pm\infty} -x^6 \end{aligned}$$

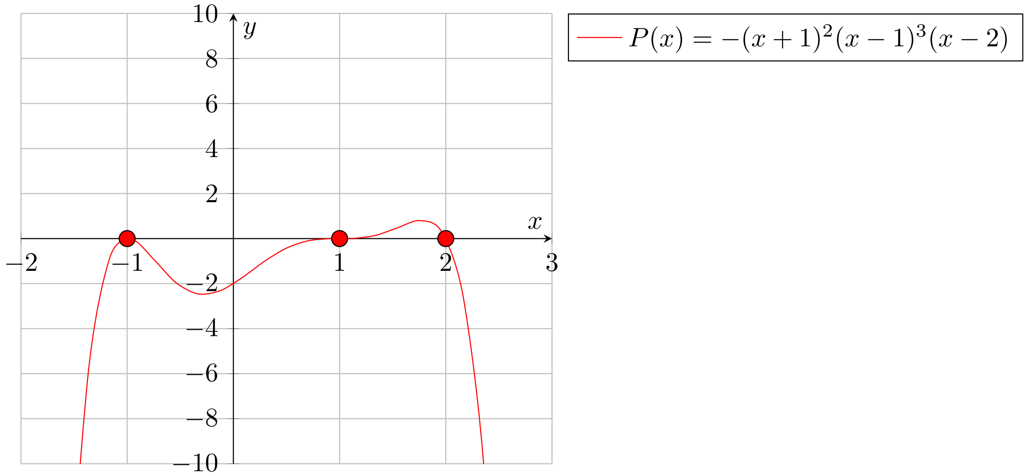
So we expect $P(x) \to -\infty$ as $x \to \infty$ and $P(x) \to -\infty$ as $x \to -\infty$.

One of the roots ($x = -1$) has multiplicity 2, which means that we do not expect a sign change. The graph will hit zero at $x = -1$ and then approach negative values again between this root and the next one. The other roots ($x = 1$ and $x = 2$) have odd multiplicity, which means that there will be a sign change. The graph will be flatter near $x = 1$ than at $x = 2$ due to the fact that the root at $x = 1$ has higher multiplicity.

We then expect:

$$\begin{aligned} x < -1 &: \ P(x) < 0 \\ -1 < x < 1 &: \ P(x) < 0 \\ 1 < x < 2 &: \ P(x) > 0 \\ x > 2 &: \ P(x) < 0 \end{aligned}$$

The graph below confirms this behavior:



- *Q32*: Factor the polynomial and use the factored forms to find the zeros. Then sketch the graph.

$$\begin{aligned} P(x) &= x^3 + 2x^2 - 8x \\ &= x(x^2 + 2x - 8) \\ &= x(x-2)(x+4) \\ \lim_{x\to\pm\infty} P(x) &= \lim_{x\to\pm\infty} x^3 \end{aligned}$$

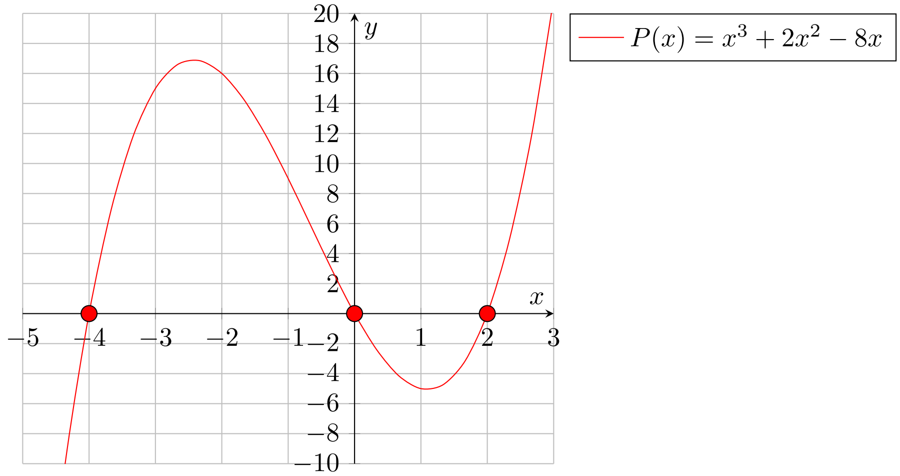
So we expect $P(x) \to \infty$ as $x \to \infty$ and $P(x) \to -\infty$ as $x \to -\infty$.

All roots have multiplicity 1, so we expect sign changes as the graph crosses each root.

Thus,

$$\begin{aligned} x < -4 &: \ P(x) < 0 \\ -4 < x < 0 &: \ P(x) > 0 \\ 0 < x < 2 &: \ P(x) < 0 \\ x > 2 &: \ P(x) > 0 \end{aligned}$$

The graph below confirms this behavior:



Section 3.3: 14,62

- *Q14*: Two polynomials P and D are given. Use either synthetic or long division to divide P(x) by D(x), and express P in the form

$$P(x) = D(x) \cdot Q(x) + R(x)$$

We are given $P(x) = 27x^5 - 9x^4 + 3x^2 - 3$ and $D(x) = 3x^2 - 3x + 1$.

Using long division:

$$\begin{array}{r|rrrrrr}
 & & & 9x^3 & +6x^2 & +3x & +2 \\
\hline
3x^2 - 3x + 1 & 27x^5 & -9x^4 & & +3x^2 & & -3 \\
 & -27x^5 & +27x^4 & -9x^3 & & & \\
\hline
 & & 18x^4 & -9x^3 & +3x^2 & & \\
 & & -18x^4 & +18x^3 & -6x^2 & & \\
\hline
 & & & 9x^3 & -3x^2 & & \\
 & & & -9x^3 & +9x^2 & -3x & \\
\hline
 & & & & 6x^2 & -3x & -3 \\
 & & & & -6x^2 & +6x & -2 \\
\hline
 & & & & & 3x & -5
\end{array}$$

Then

$$27x^5 - 9x^4 + 3x^2 - 3 = (3x^2 - 3x + 1)(9x^3 + 6x^2 + 3x + 2) + 3x - 5$$

where now $Q(x) = 9x^3 + 6x^2 + 3x + 2$ and $R(x) = 3x - 5$.

Using synthetic division is not appropriate here since the divisor is not of the form $x - c$.

- *Q62*: Show that the given values of c are zeros of $P(x)$, and find all other zeros of $P(x)$.

$$P(x) = 2x^4 - 13x^3 + 7x^2 + 37x + 15$$
$$c = -1, 3$$

Simply substitute each value of c into $P(x)$:

$$\begin{aligned}
P(-1) &= 2(-1)^4 - 13(-1)^3 + 7(-1)^2 + 37(-1) + 15 \\
&= 2 + 13 + 7 - 37 + 15 \\
&= 23 + 15 - 37 \\
&= 37 - 37 = 0 \\
P(3) &= 2(3)^4 - 13(3)^3 + 7(3)^2 + 37(3) + 15 \\
&= 81(2) - 27(13) + 63 + 111 + 15 \\
&= 162 - (270 + 81) + 189 \\
&= 162 + 189 - 351 \\
&= 351 - 351 = 0
\end{aligned}$$

How to find the remaining roots? This can be formulated as a long division problem by the product of the factors $(x - c)$:

$$\frac{2x^4 - 13x^3 + 7x^2 + 37x + 15}{(x+1)(x-3)} = \frac{2x^4 - 13x^3 + 7x^2 + 37x + 15}{x^2 - 2x - 3}$$

Use long division:

$$\begin{array}{rrrrrr}
 & & & 2x^2 & -9x & -5 \\ \hline
x^2 - 2x - 3 \big) & 2x^4 & -13x^3 & +7x^2 & +37x & +15 \\
 & -2x^4 & +4x^3 & +6x^2 & & \\ \hline
 & & -9x^3 & +13x^2 & +37x & \\
 & & 9x^3 & -18x^2 & -27x & \\ \hline
 & & & -5x^2 & +10x & +15 \\
 & & & 5x^2 & -10x & -15 \\ \hline
 & & & & & 0
\end{array}$$

So our remainder is zero (as expected) and our quotient is $Q(x) = 2x^2 - 9x - 5$. We just need to find the roots of this quadratic polynomial, and we will have found the remaining roots of $P(x)$:

$$2x^2 - 9x - 5 = (2x + 1)(x - 5)$$
$$x = \left\{5, -\frac{1}{2}\right\}$$

Thus,

$$2x^4 - 13x^3 + 7x^2 + 37x + 15 = (x + 1)(x - 3)(x - 5)(2x + 1)$$

and the roots are $\left\{-\frac{1}{2}, -1, 3, 5\right\}$.