Algebra and Calculus: Homework 5 Solutions

Section 2.4: 20,22

- *Q20*: Determine the net change and average rate of change of the function between the values of x given:

$$f(x) = 1 - 3x^2$$
$$x = 2, x = 2 + h$$

The net change is given by $\Delta y = f(b) - f(a)$, where $b > a$. So

$$\begin{aligned}\Delta y &= f(h+2) - f(2)\\ &= (1 - 3[h+2]^2) - (1 - 3(2)^2)\\ &= (1 - 3[h^2 + 4h + 4]) - (1 - 12)\\ &= -3h^2 - 12h - 11 + 11\\ &= -3h^2 - 12h\end{aligned}$$

The average rate of change equals the net change in $y = f(x)$ divided by the net change in x (that is, $b - a$):

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{-3h^2 - 12h}{(2+h) - 2}\\ &= \frac{-3h^2 - 12h}{h}\\ &= -3h - 12\end{aligned}$$

- *Q22*: Determine the net change and average rate of change of the function between the values of x given:

$$g(x) = \frac{2}{x+1}$$
$$x = 0, x = h$$

The net change is given by $\Delta y = f(b) - f(a)$, where $b > a$. So

$$\begin{aligned}\Delta y &= g(h) - g(0)\\ &= \frac{2}{h+1} - 2\\ &= \frac{2 - 2(h+1)}{h+1}\\ &= \frac{2 - 2h - 2}{h+1}\\ &= \frac{-2h}{h+1}\end{aligned}$$

The average rate of change equals the net change in $y = f(x)$ divided by the net change in x (that is, $b - a$):

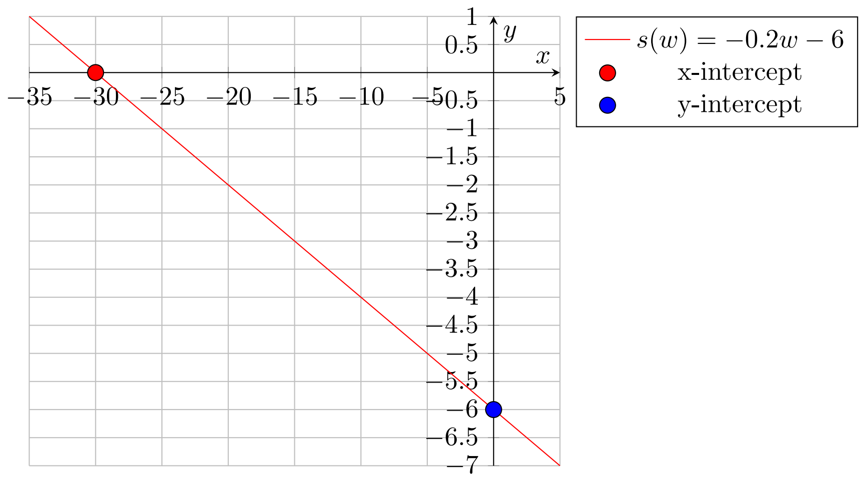
$$
\begin{aligned}
\frac{\Delta y}{\Delta x} &= \frac{\frac{-2h}{h+1}}{h - 0} \\
&= \frac{-2h}{h(h+1)} \\
&= \frac{-2}{h+1}
\end{aligned}
$$

Section 2.5: 22

- *Q22*: Sketch the graph, find the slope, and find the rate of change of the function:

$$s(w) = -0.2w - 6$$

The graph looks like:



The slope of the graph is clear by the form of the equation ($y = mx + b$), and $m = -0.2 = -\frac{1}{5}$. We can also take the x-intercept (-30,0) and y-intercept (0,-6) as our two points and calculate the slope from the graph:

$$
\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{0 - (-6)}{-30 - 0} \\
&= \frac{6}{-30} = -\frac{1}{5}
\end{aligned}
$$

The slope of the graph and the rate of change of the function are identical. Thus, the rate of change of the function is $-\frac{1}{5}$.

Section 2.6: 52

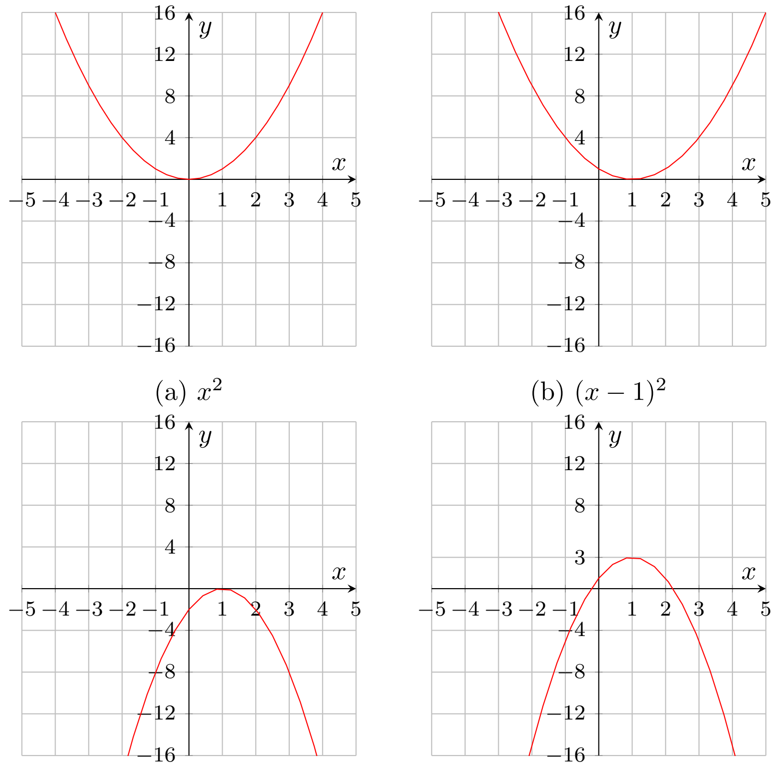
- *Q52*: Graph the function by applying transformations to the graph of a standard function:

$$y = 3 - 2(x-1)^2$$

We can take our standard function to be x^2 since we know how to graph this. The steps:

1. Graph x^2
2. Shift to the right one space $\rightarrow (x-1)^2$
3. Double the height of the graph at each point (i.e. double the value of the function), then flip about the x-axis $\rightarrow -2(x-1)^2$
4. Add three to the value at each coordinate $\rightarrow 3 - 2(x-1)^2$.

The group plot below shows each step:



(a) x^2 (b) $(x-1)^2$ (c) $-2(x-1)^2$ (d) $3-2(x-1)^2$

Section 2.7: 16,32

- *Q16*: Find $f+g$, $f-g$, fg and $\frac{f}{g}$ and their domains for

$$f(x) = \frac{2}{x+1}$$
$$g(x) = \frac{x}{x+1}$$

We see that f and g are both valid for the domain $\{x \in \mathbb{R} | x \neq -1\}$ (that is, all real numbers except -1). Then the domain for $f+g$, $f-g$ and fg must be the same as that of f and g.

The domain of $\frac{f}{g}$, however, is that of f and g, minus the values of x where $g(x) = 0$. In this case, $\frac{x}{x+1} = 0$ when $x = 0$. Thus the domain is $\{x \in \mathbb{R} | x \neq \{-1, 0\}\}$.

Here are the results:

$$f+g = \frac{x+2}{x+1}$$
$$f-g = \frac{x-2}{x+1}$$
$$fg = \frac{2x}{(x+1)^2}$$
$$\frac{f}{g} = \frac{\frac{2}{x+1}}{\frac{x}{x+1}} = \frac{2}{x}$$

One might ask why $\frac{f}{g}$ excludes $x = -1$ from the domain, since the (x+1) terms in the denominator of f and g cancel. To justify this, note that the equality

$$\frac{\frac{2}{x+1}}{\frac{x}{x+1}} = \frac{2}{x}$$

is contingent on the fact that $x \neq -1$. Let's substitute $x = -1$ into the above expression:

$$\frac{\frac{2}{(-1)+1}}{\frac{-1}{(-1)+1}} = \frac{2}{-1}$$
$$\frac{\frac{2}{0}}{\frac{-1}{0}} = -2$$
$$-2\frac{0}{0} = -2$$

and dividing by zero is an illegal operation. Thus, the domain still excludes $x = -1$.

- *Q32*: Composition of functions, $f(x) = 2x - 3$, $g(x) = 4 - x^2$.

Part (a) asks us to calculate $(f \circ f)(x)$ and part (b) asks us to calculate $(g \circ g)(x)$. Let's proceed:

$$(f \circ f)(x) = 2(2x-3) - 3$$
$$= 4x - 6 - 3 = 4x - 9$$
$$(g \circ g)(x) = 4 - (4 - x^2)^2$$
$$= 4 - (x^2 - 4)^2 = 4 - (x^4 - 8x^2 + 16)$$
$$= -x^4 + 8x^2 - 12$$