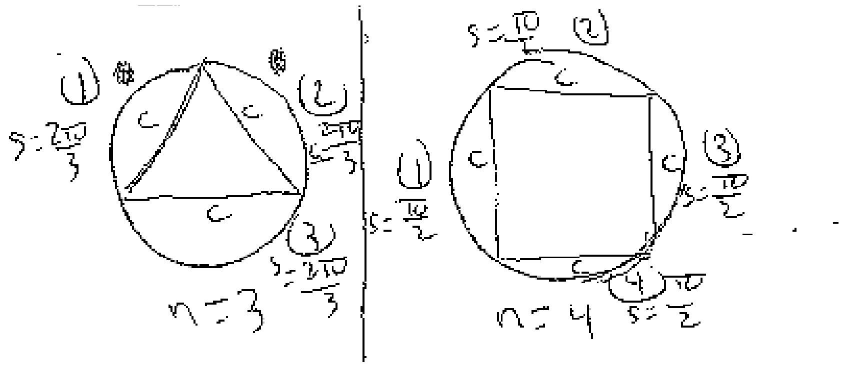
Perimeter of the n-sided regular polygon inscribed in the unit circle



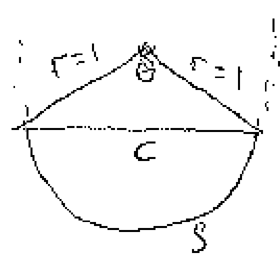
$r = 1$ (unit circle)

c = length of the chord formed by one side of the polygon

The polygon will divide the circle into n arcs of equal length. The circumference of a circle is $2\pi r$ and $r = 1$.

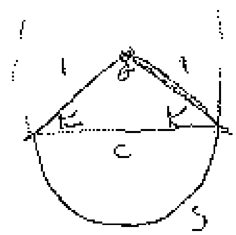
That means each arc has a length of $\frac{2\pi}{n}$ (call this S).

Consider the triangle formed by extending two lines from the center of the circle, with one line going to one end of the chord of length c, and the other line going to the other end



Since we've extended from the center of the circle to its boundaries, those two line segments have length 1, the radius of the circle. We would like to find c, the length of the chord, since

$$\boxed{P(n) = n \cdot c}$$



By the way we've drawn the line segments, the two missing angles of the triangle are equal (let's call them K). In any triangle, the angles must add up to 180°, or π. So

$$2K + \theta = \pi \Rightarrow K = \frac{\pi - \theta}{2}$$

And we can also find θ using the formula

$$S = r\theta \Rightarrow S = \theta$$

where S = the length of the arc $= \frac{2\pi}{n}$

So $\theta = \frac{2\pi}{n}$, which means $K = \frac{\pi}{2} - \frac{\pi}{n}$

Now, use the law of sines to find c

$$\frac{\sin(\theta)}{c} = \frac{\sin(K)}{1}$$

$$\frac{\sin(2\pi/n)}{c} = \frac{\sin\left(\frac{\pi}{2} - \frac{\pi}{n}\right)}{1} \Rightarrow c = \frac{\sin(2\pi/n)}{\sin\left(\frac{\pi}{2} - \frac{\pi}{n}\right)}$$

Just two more steps!

1) $\sin\left(\frac{2\pi}{n}\right) = 2\sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right)$ (using $\sin(2x) = 2\sin x \cos x$)

2) $\sin\left(\frac{\pi}{2} - \frac{\pi}{n}\right) = \underbrace{\sin\left(\frac{\pi}{2}\right)}_{1}\cos\left(\frac{\pi}{n}\right) - \underbrace{\cos\left(\frac{\pi}{2}\right)}_{0}\sin\left(\frac{\pi}{n}\right) = \cos\left(\frac{\pi}{n}\right)$

$(\sin\frac{\pi}{2} = 1)$ $(\cos\frac{\pi}{2} = 0)$

So $$c = \frac{2\sin\left(\frac{\pi}{n}\right)\cancel{\cos\left(\frac{\pi}{n}\right)}}{\cancel{\cos\left(\frac{\pi}{n}\right)}} = 2\sin\left(\frac{\pi}{n}\right)$$

Finally since $P(n) = c \cdot n$, $\boxed{P(n) = 2n\sin\left(\frac{\pi}{n}\right)}$