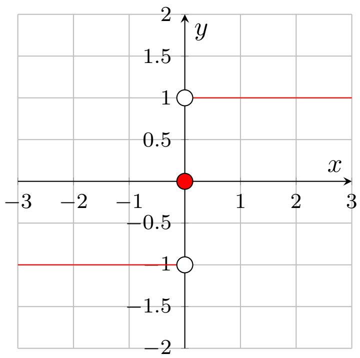
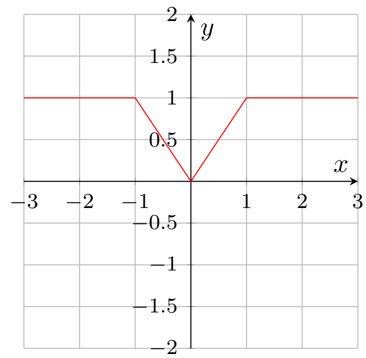
Algebra and Calculus Worksheet 5 (10-19-15)

Name: ____________________________

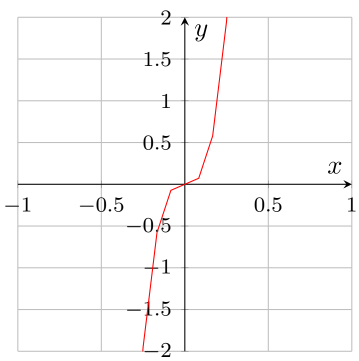
1. Which of the following are one-to-one functions?



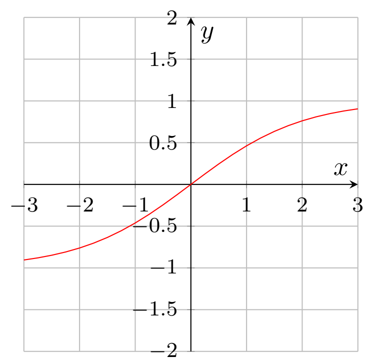
(a)



(b)



(c)



(d)

Solution: By using the horizontal and vertical line tests (all the graphs pass the vertical line test, but good to check!), we see that (c) and (d) are one-to-one functions.

2. **True or false**: An even function can never be a one-to-one function.

Solution: This is a bit of a trick question. An even function is defined by

$$f(-x) = f(x)$$

If we choose not to restrict the natural domain of the function, then the statement is true. Consider $x = c$, where $c > 0$. Then $c \neq -c$, and for an even function

$$f(-c) = f(c)$$

and so we've found an example of two distinct values of x with the same value of y. This would fail the horizontal line test.

However, if we restrict the domain to only positive values, for example, then the function **is** one-to-one; every possible input gives a unique output. The graph of this would also pass the horizontal line test.

Another example of an even function: say the domain is restricted to one point $x = 0$ and the output is given by $f(0) = c$. Then there is only one input and one output (the range is given by $y = c$). This is a one-to-one function, and it also satisfies the definition of an even function:

$$f(0) = f(-0) = c$$

3. Let b be the y-intercept of the function $f(x) = (x-2)^3 + 1$. What is $f^{-1}(b)$?

   Solution: There are two ways to do this question. The easiest way is to consider the definition of the y-intercept:

$$\begin{aligned} b &= f(0) \\ \implies 0 &= f^{-1}(b) \end{aligned}$$

   We are able to do so because the function $f(x) = (x-2)^3 + 1$ has an inverse. So the answer is zero.

   We can also (1) find the y-intercept, (2) find the inverse function and (3) substitute the y-intercept into the function. First, the y-intercept:

$$\begin{aligned} b = f(0) &= (0-2)^3 + 1 \\ &= -8 + 1 = -7 \end{aligned}$$

   Now find the inverse function:

$$\begin{aligned} y &= (x-2)^3 + 1 \\ (x-2)^3 &= y - 1 \\ x - 2 &= (y-1)^{\frac{1}{3}} \\ x &= (y-1)^{\frac{1}{3}} + 2 \end{aligned}$$

   So the inverse function is $g(x) = (x-1)^{\frac{1}{3}} + 2$. Let's substitute the y-intercept:

$$\begin{aligned} g(b) &= g(-7) \\ &= (-7-1)^{\frac{1}{3}} + 2 \\ &= (-8)^{\frac{1}{3}} + 2 \\ &= -2 + 2 \\ &= 0 \end{aligned}$$

   So we get the answer we expect.

4. Give me an example of:

   (a) A smooth, continuous function

   (b) A continuous function with a corner

   (c) A continuous function with a cusp

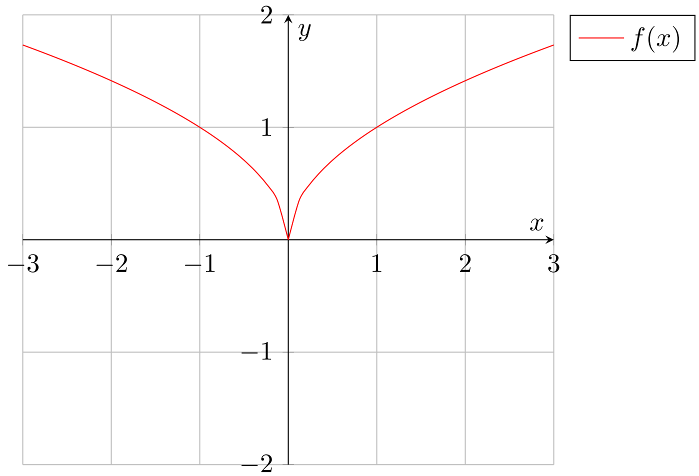
   (d) A discontinuous function

Solution: We can look at question 1 for examples of 3 out of 4 of these categories.

- (d) gives an example of a smooth continuous function
- (b) gives an example of a continuous function with a corner
- (a) gives an example of a discontinuous function

As for (c), an example of a continuous function with a cusp, consider the function $y = \sqrt{|x|}$.

The graph is below:



There is a cusp at the origin.

5. Consider the following general form of a polynomial: $P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$.

   (a) What is the leading coefficient?

   (b) Identify the constant term.

   (c) If I wanted to find the end behavior of the polynomial, which term should I look at?

   (d) What is the value of the polynomial when x=0?

   Solution:

   (a) The leading coefficient is the coefficient multiplying the term that is of highest degree. That is, a_n. We should of course assume that $a_n \neq 0$ in this example (the order of the polynomial is determined by the term of highest degree with a nonzero coefficient).

   (b) The constant term is given by the term that isn't dependent on x. That is, a_0.

   (c) To find the end behavior of the polynomial (that is, when $x \to \pm\infty$), we look at the term of highest degree. One way to see this: consider the ratio $R(x)$ between the term of highest degree and any other term $a_k x^k$, with $a_k \neq 0$ and $0 \leq k < n$:

$$\begin{aligned} R(x) &= \frac{a_n x^n}{a_k x^k} \\ &= \frac{a_n}{a_k} x^{n-k} \end{aligned}$$

Since $k < n$, $n - k > 0$. Thus, as $x \to \pm\infty$, $|x^{n-k}| \to \infty$. So this means that the term $a_n x^n$ becomes much larger in magnitude than any of the other terms as $x \to \pm\infty$. Thus, we only need look at $a_n x^n$ to determine the end behavior.

(d) When we substitute $x = 0$ into the polynomial, we get a_0. So for *small* values of x, one can deduce that a_0 becomes a relatively important term.

6. As the degree of a polynomial increases, the graph becomes **flatter** (*flatter, steeper*) around the origin and **steeper** (*flatter, steeper*) elsewhere.

7. (*Section 3.2: Q5)* Sketch a graph of the following via the *transformation of monomials* method (i.e. sketch the original monomial, and then transform it in the appropriate way):

   (a) $P(x) = x^3 - 8$

   (b) $Q(x) = -x^3 + 27$

   (c) $R(x) = -(x + 2)^3$

   (d) $S(x) = -\frac{1}{2}(x - 1)^3 + 4$

   Solution: See the plots below. The steps:

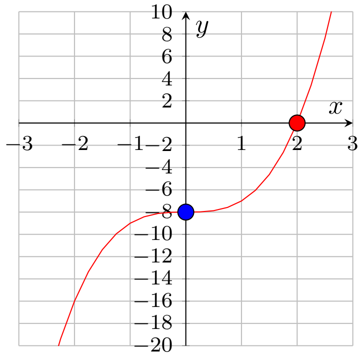
   (a) $P(x)$: Take x^3, move the graph down 8 spaces.

   (b) $Q(x)$: Take x^3, flip the graph upside down and then 27 spaces up.

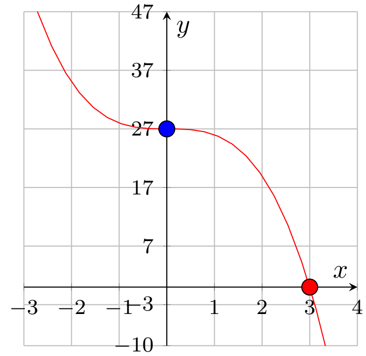
   (c) $R(x)$: Take x^3, move it two spaces to the left and then flip upside down.

   (d) $S(x)$: Take x^3, move it one space to the right, then shrink vertically by a factor of a half, then flip upside down, and finally move four spaces up.

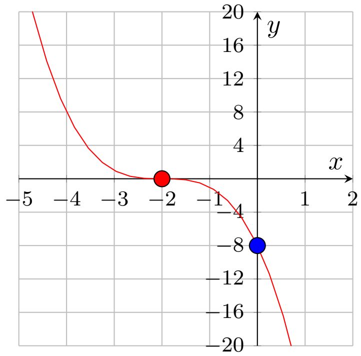
   The x-intercept is marked red and the y-intercept is marked blue.



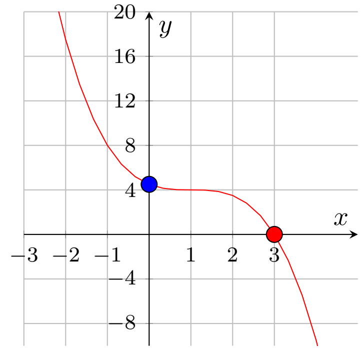
$P(x) = x^3 - 8$



$Q(x) = -x^3 + 27$



$R(x) = -(x+2)^3$



$S(x) = -\frac{1}{2}(x-1)^3 + 4$

8. (*Section 3.2: Q11)* Describe the end behavior of $R(x) = -x^5 + 5x^3 - 4x$

   Solution: We only need look at the leading term: $-x^5$. Let's see what happens when $x \to \pm\infty$:

   - As $x \to \infty$, $-x^5 \to -\infty$.
   - As $x \to -\infty$, $-x^5 \to +\infty$.

9. (*Section 3.2: Q17)* Sketch a graph of $P(x) = -x(x-3)(x+2)$.

   Solution: The function is represented as a product of factors of multiplicity 1, and so we expect a sign change at each of the roots of the equation $-x(x-3)(x+2) = 0$. The roots are

   $$x = \{0, 3, -2\}$$

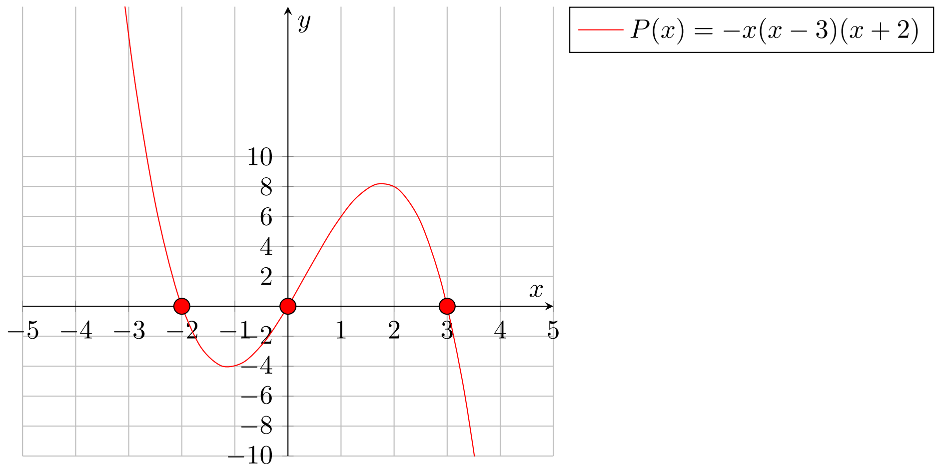
   We can look at the end behavior to see what values the graph takes on as x approaches $\pm\infty$.

   $$\lim_{x\to\pm\infty} P(x) = \lim_{x\to\pm\infty} -x^3$$
   $$\lim_{x\to-\infty} -x^3 = +\infty$$
   $$\lim_{x\to+\infty} -x^3 = -\infty$$

Thus we can break down the signs:

$$\begin{aligned} x < -2 &: \ P(x) > 0 \\ -2 < x < 0 &: \ P(x) < 0 \\ 0 < x < 3 &: \ P(x) > 0 \\ x > 3 &: \ P(x) < 0 \end{aligned}$$

The graph below confirms this behavior:



10. (*Section 3.3: Q9)* Divide $D(x)$ into $P(x)$ using both long and synthetic division, and express $P(x)$ as $P(x) = Q(x)D(x) + R(x)$.

   - $P(x) = -x^3 - 2x + 6$, $D(x) = x + 1$

   Solution: Using long division,

$$\begin{array}{r} -x^2 \quad +x-3 \\ x+1 \overline{) -x^3 \qquad\quad -2x+6} \\ \underline{x^3+x^2} \qquad\qquad \\ x^2-2x \quad \\ \underline{-x^2 \quad -x} \quad \\ -3x+6 \\ \underline{3x+3} \\ 9 \end{array}$$

   So $-x^3 - 2x + 6 = (-x^2 + x - 3)(x + 1) + 9$, with $Q(x) = -x^2 + x - 3$ and $R(x) = 9$.

   Using synthetic division:

-1	-1	0	-2	6
	↓	1	-1	3
	-1	1	-3	9

   So the coefficients we get are -1 on x^2, 1 on x, -3 as the constant term and a remainder of 9. This agrees with the long division result.

11. **True or false** If I tell you that $P(x)$ is a polynomial of degree n and $P(c) = 0$, then $S(x) = \dfrac{P(x)}{x-c}$ is a polynomial of degree n-1.

Solution: This statement is **true**. If $P(c) = 0$, then c is a root of the polynomial $P(c)$. By the factor theorem, $x - c$ is a factor of $P(x)$, or:

$$P(x) = Q(x)(x - c)$$

where $Q(x)$ is a polynomial. Because $Q(x)$ is a polynomial and we are multiplying it by a factor of $x - c$ to get $P(x)$, which is a polynomial of degree n, it follows that $Q(x)$ must be a polynomial of degree n-1.

12. (*Section 3.3: Q27)* Find the quotient using synthetic division: $\dfrac{3x^2 + x}{x + 1}$.

    Solution:

-1	3	1	0
	↓	-3	2
	3	-2	2

    So we have that $P(x) = (x + 1)(3x - 2) + 2$. In this case, $Q(x) = 3x - 2$ and $R(x) = 2$.

13. (*Section 3.3: Q57)* Use the factor theorem to show that the given value of c is a zero of $P(x)$ and find all other zeros of $P(x)$: $P(x) = x^3 + 2x^2 - 9x - 18$, $c = -2$.

    Solution: By the factor theorem, c is a zero of $P(x)$ if and only if $P(x)$ is divisible by $x - c = x + 2$. Let's confirm (we'll use long division):

$$\begin{array}{r|l} & x^2 \qquad\quad - 9 \\ \hline x+2 & x^3 + 2x^2 - 9x - 18 \\ & -x^3 - 2x^2 \\ \hline & \qquad\qquad -9x - 18 \\ & \qquad\qquad\ \ 9x + 18 \\ \hline & \qquad\qquad\qquad\quad 0 \end{array}$$

    Since the remainder is 0, $P(x)$ is indeed divisible by $x + 2$. The quotient is $x^2 - 9$, making the other roots $x = \{-3, 3\}$. Thus in total the three roots are $x = \{-3, -2, 3\}$.