Algebra and Calculus: Quiz 7 (Solutions)

Name/NetID: ____________________________

Complete all problems.

1. For **multiple choice** problems, circle the letter corresponding to the correct answer.
2. For **true or false** problems, indicate whether you believe the statement is $\boxed{\text{true}}$ or $\boxed{\text{false}}$ and put a box around your answer (as shown).
3. For **free response** problems, **show all work** and put a $\boxed{\text{box}}$ around your final answer.

Good luck!

Answers: true, true, (a), (e), (c), (b), (a), $\log(3x+1)+\frac{1}{2}\log(x+1)+\frac{1}{2}\log(x-1)-\log(x+2)$

For the first two questions, answer **true or false**

1. $\log_a(b) = x \implies b = a^x$

   Solution: This is $\boxed{\text{true}}$. It is simply the definition of a logarithm as the inverse of an exponential.

2. The range of e^x is the same as the domain of $\ln(x)$.

   Solution: This is also $\boxed{\text{true}}$. The exponential and the natural logarithm are inverse functions, so the range of the exponential must be the same as the domain of the natural logarithm.

The next five questions are multiple choice.

3. What is the value of $\log_{10}\left(\frac{1}{100}\right)$?

   (a) -2

   (b) -1

   (c) 0

   (d) 1

   (e) 2

   Solution: The answer is $\boxed{\text{(a)}}$. Proceed as follows:

$$\begin{aligned}\log_{10}\left(\frac{1}{100}\right) &= \log_{10}\left(\frac{1}{10^2}\right)\\ &= \log_{10}\left(10^{-2}\right)\\ &= -2\end{aligned}$$

   since the function $\log_{10}(x)$ is the inverse of 10^x.

4. If $\log_2(x) = \log_4(x^n)$ for all $x > 0$, what is n?

   (a) -2

   (b) -1

   (c) 0

   (d) 1

   (e) 2

   Hint: Pick a "smart" value of x and use the fact that $\log_a(a) = 1$.

   Solution: One way to do this is to use the fact that $\log_a(a) = 1$ as per the hint. Because we're using logs of base 2 and 4,

$$\log_2(2) = 1$$
$$\log_4(4) = 1$$

   A smart value of x to test is $x = 2$, since we are told that $\log_2(x) = \log_4(x^n)$. Using the given equation and the property of logs discussed earlier,

$$\log_2(2) = \log_4(2^n) = 1$$

   In order to have $\log_4(2^n) = 1$, it must be true that $2^n = 4$ (from the same property discussed earlier). Thus it follows that $n = 2$ and the answer is $\boxed{\text{(e)}}$.

   Another way to do this is to convert these to exponential form. Say they share the common value c. Then

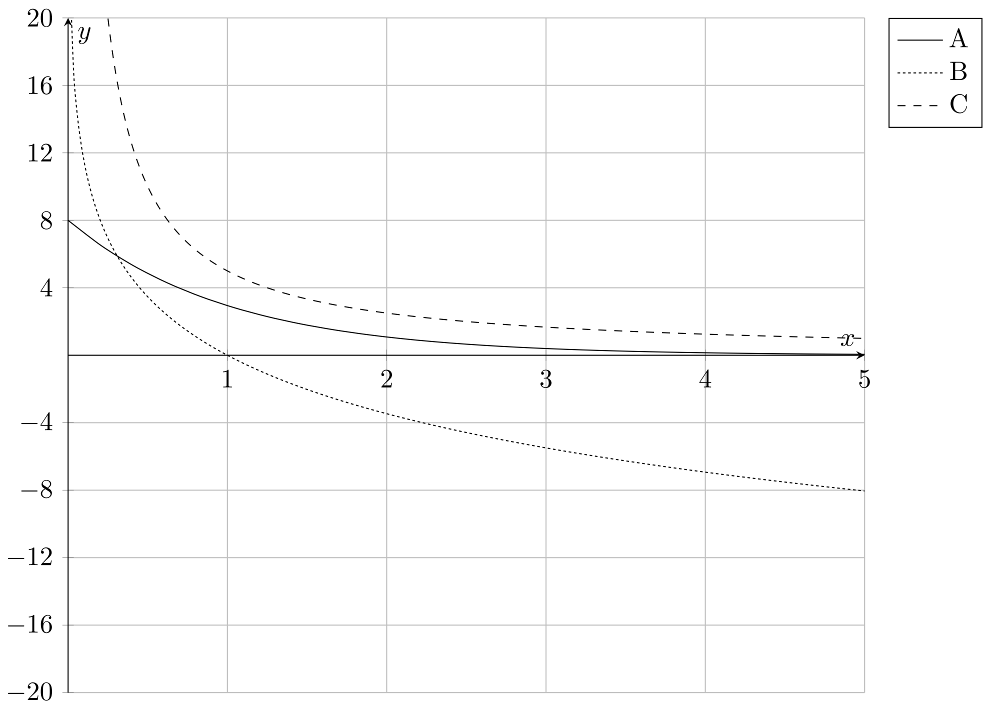
$$\log_2(x) = c \implies x = 2^c$$
$$\log_4(x^n) = c \implies x^n = 4^c = \left(2^2\right)^c = 2^{2c}$$

   Comparing the first and second equations, we see that if we take the first equation to the nth power and compare it to the second,

$$x^n = 2^{nc}$$
$$x^n = 2^{2c}$$
$$\implies nc = 2c$$
$$\implies n = 2$$

   So the answer is $\boxed{\text{(e)}}$ once again.

5. Below is a graph of three functions, with each given one of the labels A, B, or C.



The functions are

- $f(x) = -5\ln(x)$
- $g(x) = 8e^{-x}$
- $h(x) = \frac{5}{x}$

Which labels correspond to which functions?

(a) $A = f(x)$, $B = g(x)$, $C = h(x)$

(b) $A = h(x)$, $B = g(x)$, $C = f(x)$

(c) $A = g(x)$, $B = f(x)$, $C = h(x)$

(d) $A = f(x)$, $B = h(x)$, $C = g(x)$

(e) $A = g(x)$, $B = h(x)$, $C = f(x)$

Solution: The answer is $\boxed{\text{(c)}}$. The key here is to look for properties that distinguish the three given functions from each other. The things to look for:

- The function $f(x) = -5\ln(x)$ evaluates to 0 at $x = 1$ and will attain negative values for $x > 0$. This tells me that graph B must correspond to this function. From here we already know that the answer is $\boxed{\text{(c)}}$.
- The function $8e^{-x}$ evaluates to 8 at $x = 0$ and asymptotes at $y = 0$. The graph has to be A, since this is the only graph that hits the y-axis (and at $y = 8$).
- The graph $h(x) = \frac{5}{x}$ asymptotes at both the x and y axis, and the only graph that does this is C

6. What is the domain of $f(x) = \dfrac{\log(5-x)}{\sqrt{x^2-1}}$?

   (a) $[1, 5]$

   (b) $(-\infty, -1) \cup (1, 5)$

   (c) $[1, 5)$

   (d) $(-\infty, -1] \cup [1, 5)$

   (e) $(5, \infty)$

   Solution: The answer is $\boxed{\text{(b)}}$. We need to examine the numerator and denominator and identify the regions where each of these is valid, and take the intersection, also remembering to exclude the value of x that sets the denominator equal to 0:

$$\log(5-x) \to x < 5$$
$$\frac{1}{\sqrt{x^2-1}} \to x^2 > 1 \implies |x| > 1$$

   Taking the intersection of the two domains, we see that we need $1 < x < 5$ and $x < -1$ for the input to $f(x)$ to be valid. This corresponds to choice $\boxed{\text{(b)}}$.

7. What are the horizontal asymptotes of $\dfrac{e^{-x}}{2e^{-x}+1}$?

   (a) $y \to \frac{1}{2}$ as $x \to -\infty$ and $y \to 0$ as $x \to \infty$

   (b) $y \to 0$ as $x \to -\infty$ and $y \to \frac{1}{2}$ as $x \to \infty$

   (c) $y \to \frac{1}{2}$ as $x \to \pm\infty$

   (d) $y \to 0$ as $x \to \pm\infty$

   (e) $y \to 2$ as $x \to -\infty$ and $y \to 0$ as $x \to \infty$

   Hint: What is the end behavior of e^{-x}?

   Solution: The key is to understand the end behavior of e^{-x}. One should know that for $f(x) = e^x$, the function approaches infinity as $x \to \infty$ and it approaches 0 as $x \to -\infty$ (the range is $(0, \infty)$). This is switched for $f(-x) = e^{-x}$. That is, $f(-x) \to 0$ as $x \to \infty$ and $f(-x) \to \infty$ as $x \to -\infty$.

   Let $b = e^{-x}$. Then we can write the expression given in the above question as

$$\frac{b}{2b+1}$$

   We know that as $x \to \infty$, $b \to 0^+$. So as $b \to 0$,

$$\frac{b}{2b+1} \to \frac{0}{2(0)+1} = 0$$

   It follows that the horizontal asymptote as $x \to \infty$ is at $y = 0$.

As $x \to -\infty$, $b \to \infty$, so the leading term in the numerator and denominator is given by b. Using what we know about end behavior, as $b \to \infty$,

$$\frac{b}{2b+1} \to \frac{b}{2b} = \frac{1}{2}$$

So the horizontal asymptote as $x \to -\infty$ is at $y = \frac{1}{2}$. Thus the answer is $\boxed{\text{(a)}}$.

The next question is a free response question.

8. Use the laws of logarithms to expand the following expression as completely as possible:

$$\log\left((3x+1)\sqrt{\frac{x^2-1}{x^2+4x+4}}\right)$$

Solution: First, separate the products using the fact that logs of products are equal to sums of logs:

$$\log(3x+1) + \log\left(\sqrt[2]{\frac{x^2-1}{x^2+4x+4}}\right)$$

The first term is as simple as possible. As for the second, let's first rewrite the square root using fractional powers:

$$\sqrt[2]{\frac{x^2-1}{x^2+4x+4}} = \left(\frac{x^2-1}{x^2+4x+4}\right)^{\frac{1}{2}}$$

Let's use the property of the log that allows us to move powers from the exponent to the outside of the log:

$$\log\left(\left(\frac{x^2-1}{x^2+4x+4}\right)^{\frac{1}{2}}\right) = \frac{1}{2}\log\left(\frac{x^2-1}{x^2+4x+4}\right)$$

Now we can factor the numerator and denominator of the second term:

$$\frac{1}{2}\log\left(\frac{x^2-1}{x^2+4x+4}\right) = \frac{1}{2}\log\left(\frac{(x+1)(x-1)}{(x+2)^2}\right)$$

Use the fact that logs of quotients are differences of logs:

$$\frac{1}{2}\log\left(\frac{(x+1)(x-1)}{(x+2)^2}\right) = \frac{1}{2}\left(\log((x+1)(x-1)) - \log\left((x+2)^2\right)\right)$$

Lastly we separate products in the first term and drop the power to the front of the log in the second term:

$$\frac{1}{2}\left(\log((x+1)(x-1)) - \log\left((x+2)^2\right)\right) = \frac{1}{2}\left(\log(x+1) + \log(x-1) - 2\log(x+2)\right)$$

So our final answer is

$$\log(3x+1) + \frac{1}{2}\log(x+1) + \frac{1}{2}\log(x-1) - \log(x+2)$$