Algebra and Calculus: Quiz 5 (Solutions)

Name/NetID: ____________________________

Complete all problems.

1. For **multiple choice** problems, circle the letter corresponding to the correct answer.
2. For **true or false** problems, indicate whether you believe the statement is true or false and put a box around your answer (as shown).
3. For **free response** problems, **show all work** and put a box around your final answer.

Good luck!

Answers: False, False, True, C, A, B, $x = \{-1, 1\}$

1. For the following three problems, answer **true or false**:

   (a) The function $f(x) = (x-1)^2(x+3)(x-4)^3$ changes sign three times.

   Solution: False. We see the roots of the polynomial are $x = \{-3, 1, 4\}$. Consider the multiplicity of each term:

   - $x + 3$ has odd multiplicity (1), and so the function will change sign between $x < -3$ and $-3 < x < 1$.
   - $(x-1)^2$ has even multiplicity (2), and so the function will not change sign between $-3 < x < 1$ and $1 < x < 4$.
   - $(x+4)^3$ has odd multiplicity (3), and so the function will change sign between $1 < x < 4$ and $x > 4$.

   Thus, the function $f(x)$ only changes sign twice.

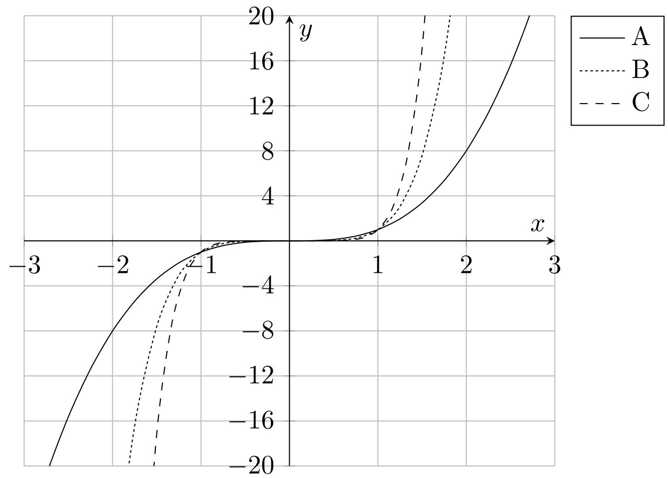
   (b) If a function has a corner, it must be discontinuous.

   Solution: False. The presence of a corner is unrelated to discontinuity. It *is*, however, related to smoothness. A function with a corner is not smooth.

   (c) The end behavior of a polynomial $a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$, $a_n \neq 0$, is determined by the term $a_n x^n$.

   Solution: True. The term of highest degree in a polynomial is what sets the behavior of the function as $x \to \pm\infty$.

2. Consider the graph of three functions below. Each is given the labels A, B, and C:



Which function corresponds to each graph?

(a) A: $(x-2)^3$, B: $(x-2)^5$, C: $(x-2)^7$

(b) A: $(x-2)^7$, B: $(x-2)^5$, C: $(x-2)^3$

(c) A: x^3, B: x^5, C: x^7

(d) A: x^7, B: x^5, C: x^3

(e) A: x^5, B: x^7, C: x^3

Solution: The answer is $\boxed{\text{(c)}}$. We know that the higher the order of a polynomial, the steeper it is away from the origin. From A to C the graph gets steeper, meaning that the polynomials must go from lowest to highest order from A to C. Thus the answer is (c).

Answers (a) and (b) are thrown in as false answers. There is nothing to indicate that the graphs of the functions x^3, x^5 and x^7 are shifted to the right two spaces.

3. You are told that $P(x) = Q(x)(x-1) + R(x)$. Given $P(x) = x^2 + 3x - 1$, what are $Q(x)$ and $R(x)$?

(a) $Q(x) = x + 4$, $R(x) = 3$

(b) $Q(x) = x + 3$, $R(x) = 4$

(c) $Q(x) = x + 2$, $R(x) = 0$

(d) $Q(x) = 3$, $R(x) = x + 4$

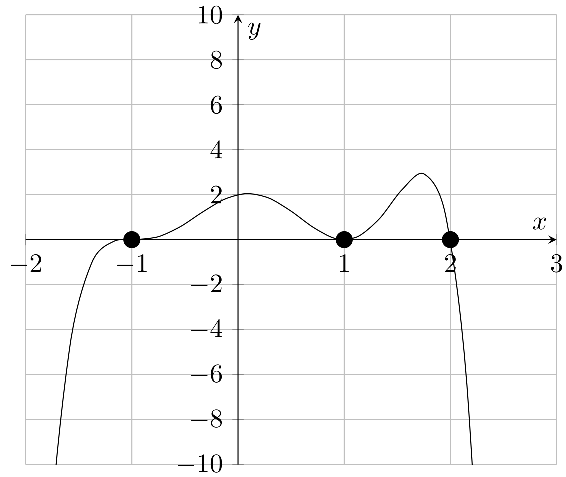
(e) $Q(x) = 4$, $R(x) = x + 3$

Solution: The answer is $\boxed{\text{(a)}}$. This is simply a division problem. We are given $P(x)$ and need to divide by $x - 1$. Using synthetic division,

1	1	3	-1
	↓	1	4
	1	4	3

So $Q(x) = x + 4$ and the remainder is $R = 3$.

4. What function is this?



(a) $f(x) = -(x+1)^2(x-1)^3(x-2)$

(b) $f(x) = -(x-1)^2(x+1)^3(x-2)$

(c) $f(x) = -(x-1)^2(x+1)^3(x-2)^2$

(d) $f(x) = -(x-1)(x+1)^3(x-2)^2$

(e) $f(x) = -(x-1)^2(x+1)^2(x-2)$

Solution: The answer is $\boxed{\text{(b)}}$. The way to solve this problem is to look at (1) the end behavior and (2) the sign changes.

We can eliminate (c) and (e) immediately because the end behavior of these looks like an odd power of x:

$$(c) \sim -x^7$$
$$(e) \sim -x^5$$

and so the sign when $x \to \infty$ will be opposite that when $x \to -\infty$. The graph, however, reflects the end behavior of an even power of x.

Now we look at the multiplicity of each term to get the sign changes. We must have odd multiplicity at $x = -1$, even multiplicity at $x = 1$ and odd multiplicity at $x = 2$. In other words, we need (x+1) and (x-2) to have odd powers and (x-1) to have an even power. Choice (b) is the only choice for which this is true.

5. You are told that two roots of the polynomial $P(x) = 2x^4 - 5x^3 - 5x^2 + 5x + 3$ are $x = -\frac{1}{2}$ and $x = 3$. Find the other roots.

Solution: This is just a polynomial long division problem. Compute the divisor:

$$D(x) = (2x+1)(x-3) = 2x^2 - 5x - 3$$

Note that we could have used $(x + \frac{1}{2})$ instead of $(2x+1)$; this will introduce a factor of two that will change the coefficients on our quotient, but will not change our answer.

Now divide $D(x)$ into $P(x)$:

$$
\begin{array}{r}
x^2 \qquad\quad - 1 \\
2x^2 - 5x - 3 \overline{\big) \; 2x^4 - 5x^3 - 5x^2 + 5x + 3} \\
\underline{-\,2x^4 + 5x^3 + 3x^2 \qquad\qquad} \\
-\,2x^2 + 5x + 3 \\
\underline{2x^2 - 5x - 3} \\
0
\end{array}
$$

So our quotient is $x^2 - 1$ with no remainder (as expected, if we divide correctly). Then from here it is clear that the roots are $x = \{-1, 1\}$.