Algebra and Calculus Worksheet: Mega Problem (Midterm 1)

Name: ____________________________

One of the purposes of the initial few weeks of lecture preceding the midterm is to drive home the point that

1. There are many ways to solve the same problem, BUT
2. Different ways of solving the same problem can lead to different insights and realizations

I'm hoping the following mega problem drives this point home by tying in pretty much everything learned until now.

The Problem

You're a scientist working in a biology laboratory and you're studying two populations of bacteria competing for resources in the same pool, subject to certain conditions (think light input, food sources, etc.). Call one of them population A (P_A), and the other population B (P_B). One of your lab mates ran a previous experiment where she measured the number of species in each population at different times and wrote them into a table. She started with equal amounts of the populations. Here are the values:

Time (hours)	**A**	**B**
0	30	30
2	22	38
4	22	38
6	54	6

1. First, what do you notice about the total number of organisms present in the pool? How does it vary with time?

   Solution: The total number remains constant. Note that 30+30 = 22+38 = 22+38 = 54+6 = 60. So when one organism decreases by a certain amount, the other grows by an equal amount. This is a common property of simple biological models where we assume matter is conserved.

2. What is the domain for the total length of the experiment? (What is the variable, and on which interval is it defined?). Name one thing that must be true about the range (*Hint: we're dealing with numbers of species*).

   Solution: Your labmate let the bacteria grow for 6 hours. So the domain is $0 \leq t \leq 6$. One thing we know about the range is that it must be greater than or equal to zero; we can't have negative numbers of bacteria!

3. Your labmate makes a note that the two populations grow according to a **cubic polynomial**, one per population. Find both polynomials. Note that you should be able to confirm your answer to question 1 using your answer to this question.

   Hint: I claim that if you give me four values at four different times, I can give you a unique cubic polynomial. We have four values at four different times, and

$$at^3 + bt^2 + ct + d = \text{Population at time t}$$

We already know d, as we simply need to plug in t=0 and we get the same number for both populations. Now we have to solve three equations for a, b, and c for each population (so two sets of three interdependent equations).

Solution: Pop A:$\frac{1}{2}t^3 - 2t^2 - 2t + 30$. Pop B: $-\frac{1}{2}t^3 + 2t^2 + 2t + 30$. We can find this by using the table. For population A:

$$(2^3)a + (2^2)b + (2)c + 30 = 8a + 4b + 2c + 30 = 22$$
$$(4^3)a + (4^2)b + (4)c + 30 = 64a + 16b + 4c + 30 = 22$$
$$(6^3)a + (6^2)b + (6)c + 30 = 216a + 36b + 6c + 30 = 54$$

We can simplify these equations:

$$8a + 4b + 2c = -8$$
$$64a + 16b + 4c = -8$$
$$216a + 36b + 6c = 24$$

We can substitute to solve for everything. From the first equation,

$$c = -4a - 2b - 4$$

Second equation:

$$64a + 16b - 4(4a + 2b + 4) = -8$$
$$48a + 8b = 8$$
$$6a + b = 1$$
$$b = 1 - 6a$$
$$c = -4a - 2(1 - 6a) - 4 = 8a - 6$$

Now the last equation:

$$216a + 36(1 - 6a) + 6(8a - 6) = 24$$
$$216a - 216a + 36 + 48a - 36 = 24$$
$$48a = 24$$
$$\implies a = \frac{1}{2}$$

So $b = 1 - 6\left(\frac{1}{2}\right) = -2$ and $c = 8\left(\frac{1}{2}\right) - 6 = -2$. And we know $d = 30$ from substituting t=0. Thus, for population A:

$$P_A = \frac{1}{2}t^3 - 2t^2 - 2t + 30$$

We can apply the same logic to P_B. It turns out that the polynomial you get is

$$P_B = -\frac{1}{2}t^3 + 2t^2 + 2t + 30$$

Note that when you add the two populations together:

$$P_A + P_B = \cancel{\frac{1}{2}t^3} \cancel{-2t^2} \cancel{-2t} + 30 \cancel{-\frac{1}{2}t^3} + \cancel{2t^2} + \cancel{2t} + 30 = 60$$

This confirms the answer you should have gotten for part (a).

4. You observe that the two populations start with the same number (30). What is the trend for each population up until t=6 hours? When is population A greater than population B? Vice versa? Use the table to start, then use the values of your functions P_A and P_B to fill in gaps (e.g. between t=2 and t=4, are the populations actually staying the same for two hours?).

   Solution: From the table, population B appears to be increasing through time t=2, while population A is decreasing. Between times t=2 and t=4, the values do not change on the table, BUT if you plug in values for t between 2 and 4 (3, for example), you'll see that that population B must have increased, and then decreased during this interval, and vice versa for A. Then after t=4, the populations continue on in the directions they were moving (population A continues increasing, population B continues decreasing).

5. From the previous question, you might have noted that at one point, one population is greater than the other, but this switches later on in the table. Thus, we suspect that at some point the two populations become equal again. Find the time where this happens.

   Hint: If the two populations are equal, then $P_A = P_B$. Manipulate the equation to solve for time.

   Solution: If we let $P_A = P_B$. Then

$$\begin{aligned}\frac{1}{2}t^3 - 2t^2 - 2t + 30 &= -\frac{1}{2}t^3 + 2t^2 + 2t + 30\\ t^3 - 4t^2 - 4t &= 0\\ t(t^2 - 4t - 4) &= 0\end{aligned}$$

   We know that t=0 is a solution (which was obvious since we started with the two populations being equal). We're left with a quadratic to solve:

$$\begin{aligned}t^2 - 4t - 4 &= 0\\ t &= \frac{4 \pm \sqrt{16 - 4(-4)}}{2}\\ &= \frac{4 \pm \sqrt{32}}{2} = \frac{4 \pm 4\sqrt{2}}{2}\\ &= 2 \pm 2\sqrt{2}\end{aligned}$$

   We have three solutions, but note that one of them is outside our domain! $t = 2 - 2\sqrt{2}$ is less than zero, so we exclude the value there from our solution set. Thus,

$$t = 2 + 2\sqrt{2}$$

6. Let $F(t) = P_B(t) - P_A(t) = -t(t^2 - 4t - 4)$ be a function representing the difference between the two populations $P_B(t)$ and $P_A(t)$. Where does this equal zero? *Hint: we've done this already.*

   Solution: We already solved the equation $P_A = P_B$, which is the same as solving the equation $P_B - P_A = 0$. Thus, we know that $F = 0$ at t=0 and $t = 2 + 2\sqrt{2}$.

7. You decide you would like a graphical representation of the difference between the two populations. Sketch the graph and shade the region where $P_B > P_A$, and write out the solution set. Confirm that this agrees with the original table you were given.

Solution: The graph of $F(t) = P_B - P_A$ is plotted below. The region you should shade is the region where the curve is greater than zero (i.e. $0 < t < 2+2\sqrt{2}$). Note that we were given a strict inequality, so we do not include the points where $F = 0$ in the solution set.

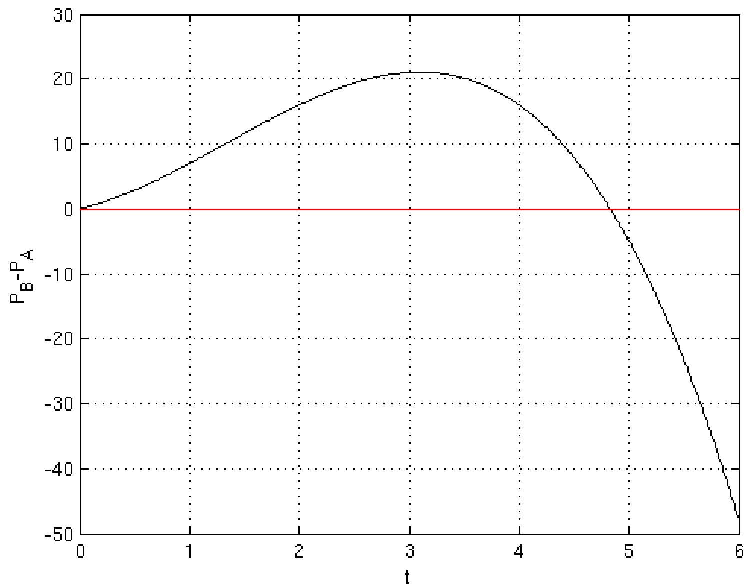

Figure 1: Graph of $P_B - P_A$.

The way to find this is to apply the sign test. We know our two regions in the domain are $0 < t < 2+2\sqrt{2}$ and $2 + 2\sqrt{2} < t < 6$. Pick a value of t in each region. If we try t=1 in the first region,

$$\begin{aligned} F(1) &= -(1)(1^2 - 4(1) - 4) \\ &= -1(1 - 4 - 4) = 7 > 0 \end{aligned}$$

and if we try t=5 in the second region, we find that

$$\begin{aligned} F(5) &= -(5)(5^2 - 4(5) - 4) \\ &= -5(25 - 20 - 4) = -5 < 0 \end{aligned}$$

Then we conclude that $P_B - P_A > 0$ in the region $0 < t < 2 + 2\sqrt{2}$.

8. Consider the two populations at the time $t = 6$. What is the **average rate** at which each population would have to grow/shrink from time $t = 0$ in order to reach its value at $t = 6$?

   Solution: We use the rate of change formula here. Luckily, all the information we need is in the table. For population 1:

$$\begin{aligned} V_A &= \frac{P_A(6) - P_A(0)}{6 - 0} = \frac{54 - 30}{6} = \frac{24}{6} = 4 \text{ per hour} \\ V_B &= \frac{P_B(6) - P_B(0)}{6 - 0} = \frac{6 - 30}{6} = \frac{-24}{6} = -4 \text{ per hour} \end{aligned}$$

   So on average, population A grows by 4 per hour and population B decreases by 4 per hour. This again agrees with the fact that the growth in one population is compensated by an equal decrease in the other population.

9. From your answer to the previous question, explain what is missed by using an average rate of change to explain about how the bacteria grow/decay with time (rely on your answers to previous questions for this). If I only sampled the populations at t=0 and t=6, I would have an entirely different idea of how the populations changed with time. Why?

   Hint: In particular, look at the graph you sketched. Would that region where the inequality is satisfied exist if I only considered the average rates of change?

   Solution: The point of this is that while the average rate of change is often an intuitive measure of how something moves (e.g. traveling from one city to another; if you give me the distance and the time, the average rate of change would give me the approximate speed I would need to travel to get there in the given amount of time), it's a measure that doesn't tell the full story. We found that our bacteria trade places as dominant species; first, P_B is greater than P_A. Then, we found the time where there is a tradeoff, and P_A becomes greater than P_B. The average rate of change only captures the differences between where you start and where you end. So it tells us that, on average, population B is decreasing while population A is increasing.