Algebra and Calculus: Homework 8 Solutions

Section 3.6: 20,40,48,56,76

Q20: $r(x) = \frac{2x-9}{x-4}$. Express as a transformation of $\frac{1}{x}$ and graph.

In general for a rational function of the form $r(x) = \frac{ax+b}{cx+d}$ (assuming $c \neq 0$), we may rewrite it as

$$\begin{aligned} r(x) &= \frac{a}{c} + \frac{b - \frac{ad}{c}}{c\left(x + \frac{d}{c}\right)} \\ &= \frac{a}{c} + \left(b - \frac{ad}{c}\right) f\left(c\left[x + \frac{d}{c}\right]\right) \end{aligned}$$

where $f(x) = \frac{1}{x}$ (this is simply the result of dividing the numerator by the denominator, check for yourself!)

The new horizontal asymptote is at $y = \frac{a}{c}$ and the new vertical asymptote is at $x = -\frac{d}{c}$. So for our problem,

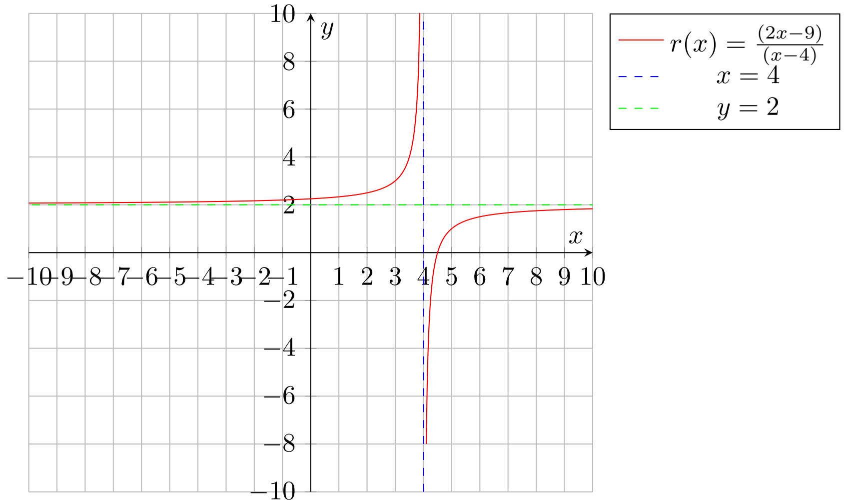
$$\begin{aligned} r(x) &= 2 - \frac{1}{x-4} \\ &= 2 - f(x-4) \end{aligned}$$

Then we see that there are three steps here:

1. Shift $\frac{1}{x}$ four units to the right
2. Reflect about the x-axis
3. Move up two units

What effect will this have on asymptotes? The vertical asymptote will now be at $x = 4$ and will mark a shift from positive to negative values (due to the reflection about the x-axis). The horizontal asymptote will now be at $y = 2$.

See the graph:



Domain: $x \neq 4$

Range: $y \neq 2$

Q40: Find vertical and horizontal asymptotes (if any) of

$$r(x) = \frac{5x^3}{x^3 + 2x^2 + 5x}$$

First, factor:

$$r(x) = \frac{5x^3}{x(x^2 + 2x + 5)}$$

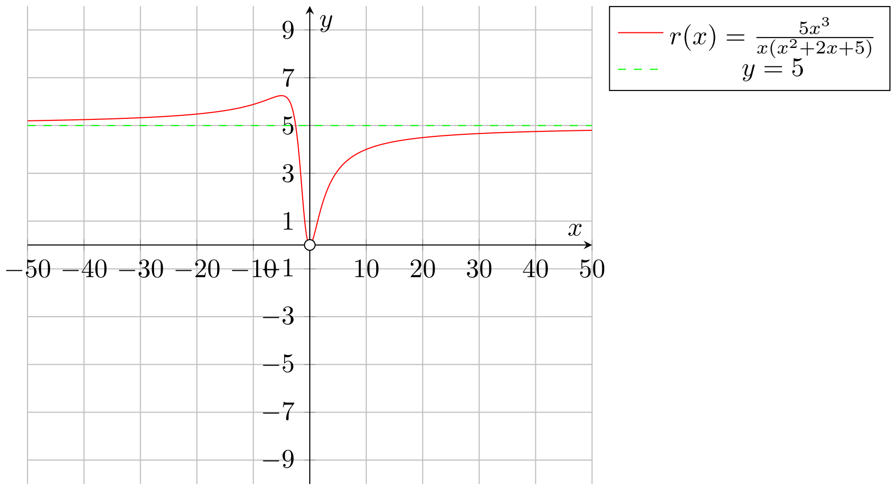
We see that at $x = 0$, our graph has a **hole**; we must leave $x = 0$ out of the domain, though there will not be a vertical asymptote there (since the factor x in the denominator will cancel with the $5x^3$ in the numerator). We can test for vertical asymptotes by setting $x^2 + 2x + 5 = 0$:

$$x = \frac{-2 \pm \sqrt{4 - 4(1)(5)}}{2}$$

But this will be complex, since the discriminant is negative. Thus, there are no vertical asymptotes.

What about horizontal asymptotes? Let's look at the end behavior. Both the numerator and denominator are like x^3 with respect to their end behavior, so the horizontal asymptote is given by the ratio between the coefficients of the x^3 terms, that is $y = 5$. So there is a horizontal asymptote at $y = 5$.

The plot confirms our findings:



Q48: Find intercepts and asymptotes, sketch a graph of rational function. State domain and range.

$$r(x) = \frac{x^2 + 2x + 3}{2x^2 + 4x + 2}$$

First factor:

$$r(x) = \frac{x^2 + 2x + 3}{2(x + 1)^2}$$

There is no x-intercept, as the discriminant of the numerator is $b^2 - 4ac = 4 - 4(1)(3) < 0$, so the roots are complex. Thus, the graph will not touch the x-axis. This also means that the numerator is always positive (check). As for the y-intercept:

$$r(0) = \frac{3}{2(1)^2} = \frac{3}{2}$$

so the y-intercept is $y = \frac{3}{2}$.

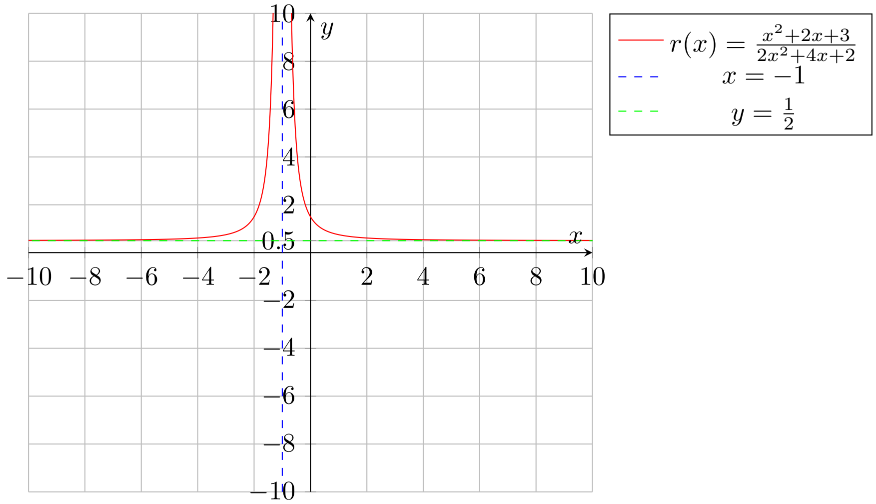
Vertical asymptotes occur whenever the denominator is equal to zero. That happens when $x = -1$. Note that in this case the vertical asymptote occurs as a **double root** of a quadratic polynomial, so the limit as $x \to -1$ will be the same from both sides:

$$\lim_{x \to -1^-} \frac{x^2 + 2x + 3}{2(x + 1)^2} = +\infty$$
$$\lim_{x \to -1^+} \frac{x^2 + 2x + 3}{2(x + 1)^2} = +\infty$$

The horizontal asymptotes are found by taking $x \to \pm\infty$. Because the numerator and the denominator are both quadratic polynomials, the asymptote will be given by the ratio between the coefficient on the numerator's quadratic term and the denominator's quadratic term:

$$y = \frac{x^2}{2x^2} = \frac{1}{2}$$

The graph confirms our findings:



Domain: $x \neq -1$

Range: $y > \frac{1}{2}$

Q56: Find intercepts and asymptotes, sketch a graph of rational function. State domain and range.

$$r(x) = \frac{3x^2 + 6}{x^2 - 2x - 3}$$

First factor:

$$r(x) = \frac{3(x^2 + 2)}{(x - 3)(x + 1)}$$

There is no x-intercept, as the discriminant of the numerator is negative, so the roots are complex. Thus, the graph will not touch the x-axis. The numerator is always positive. As for the y-intercept:

$$r(0) = \frac{6}{-3} = -2$$

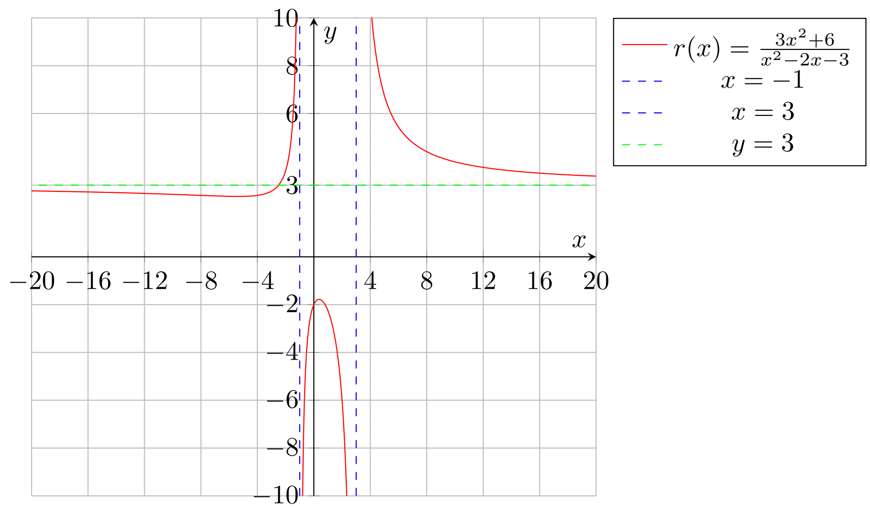
so the y-intercept is $y = -2$.

Vertical asymptotes occur whenever the denominator is equal to zero. That happens when $x = 3$ and $x = -1$. So we have a vertical asymptote at $x = 3$ and $x = -1$.

The horizontal asymptotes are found by taking $x \to \pm\infty$. Because the numerator and the denominator are both quadratic polynomials, the asymptote will be given by the ratio between the coefficient on the numerator's quadratic term and the denominator's quadratic term:

$$y = \frac{3x^2}{x^2} = 3$$

The graph confirms our findings:



From the graph we see that the domain is simply $x \neq -1$ and $x \neq 3$. What about the range? This is trickier. What is the minimum value reached for $x < -1$? It's a value slightly less than $y = 3$ according to the graph, but it's hard to see where exactly.

One way to do this: Let $r(x) = c$ for some value of x and some constant c we want to determine. What is the smallest positive value of c that satisfies this equation for real x?

$$\frac{3(x^2+2)}{(x-3)(x+1)} = c$$
$$\implies 3x^2 + 6 = cx^2 - 2cx - 3c$$
$$\implies (3-c)x^2 + (2c)x + (6+3c) = 0$$

This is a quadratic equation:

$$\begin{aligned} x &= \frac{-2c \pm \sqrt{4c^2 - 4(3-c)(6+3c)}}{2(3-c)} \\ &= \frac{-2c \pm \sqrt{4c^2 + 12(c-3)(c+2)}}{6-2c} \\ &= \frac{-2c \pm \sqrt{4c^2 + 12c^2 - 12c - 72}}{6-2c} \\ &= \frac{-2c \pm \sqrt{16c^2 - 12c - 72}}{6-2c} \end{aligned}$$

If we want $c > 0$ for real x, we also want the discriminant to be nonnegative. So we need $c > 0$ and

$16c^2 - 12c - 72 \geq 0$. Let's find the roots:

$$\begin{aligned}
16c^2 - 12c - 72 &= 0 \\
4c^2 - 3c - 18 &= 0 \\
c &= \frac{3 \pm \sqrt{9 - 4(4)(-18)}}{8} \\
&= \frac{3 \pm \sqrt{9 + 16(18)}}{8} \\
&= \frac{3 \pm \sqrt{9 + 180 + 6(18)}}{8} \\
&= \frac{3 \pm \sqrt{9 + 180 + 108}}{8} \\
&= \frac{3 \pm \sqrt{297}}{8}
\end{aligned}$$

Because $\sqrt{297} > 3$ and we need c to be positive, we see that we need the smallest possible value of c that works is $c = \frac{3+\sqrt{297}}{8} \approx 2.53$. Thus, in our range the region

$$\left[\frac{3 + \sqrt{297}}{8}, 3\right)$$

We can do the same thing for the region between $x = -1$ and $x = 3$, except now $c < 0$. What is the upper limit of that "hump"? This would be the largest negative value (i.e. the "least negative" value), which we see is given by the other root of the equation we just solved:

$$c = \frac{3 - \sqrt{297}}{8} \approx -1.78$$

This is indeed the least negative value of c that maintains $c < 0$ and $16c^2 - 12c - 72 \geq 0$ (do a sign test and check).

So our range in total is

$$\left(-\infty, \frac{3 - \sqrt{297}}{8}\right] \cup \left[\frac{3 + \sqrt{297}}{8}, 3\right) \cup (3, \infty)$$

Q76: Find the slant asymptote and vertical asymptotes:

$$r(x) = \frac{2x^3 + 2x}{x^2 - 1}$$

Let's divide:

$$\begin{array}{r|r}
 & 2x \\
\hline
x^2 - 1 & 2x^3 + 2x \\
 & -\ 2x^3 + 2x \\
\hline
 & 4x
\end{array}$$

So

$$\frac{2x^3 + 2x}{x^2 - 1} = 2x + \frac{4x}{x^2 - 1}$$

We see that as $x \to \pm\infty$ the graph behaves like $y = 2x$. We can find the vertical asymptotes by factoring the denominator:

$$\begin{aligned} x^2 - 1 &= 0 \\ \implies (x+1)(x-1) &+ 0 \\ &\implies x = \{-1, 1\} \end{aligned}$$

The graph: