Algebra and Calculus: Homework 3 Solutions

A note about solving inequalities:

Consider the example inequality $x^2 + 2x > 3$ (this is the first question in this problem set).

Note that the inequality above may also be written as $x^2 + 2x - 3 > 0$. Thus, we want to identify the regions where the polynomial on the left-hand side of the inequality is positive.

The strategy is first to find the roots of the polynomial, i.e. where it is equal to zero. The roots of the polynomial potentially divide regions of different signs. We know that $x^2 + 2x - 3 = 0$ at the roots, which we will need to solve for. For this problem the roots happen to be $x = \{-3, 1\}$. Then for values of x between the roots, or away from the roots in general, the polynomial will either take on positive or negative values. For this inequality, we want to identify the regions where $x^2 + 3x - 3$ is positive.

That is where the sign test comes in. We test values of x between the roots, less than the smallest root, and larger than the largest root. We only need to test one value between roots to learn the sign of a function within the region between the roots (think about why this is true).

In this case, the solution set consists of those regions with positive values. Were the inequality $x^2+2x < 3$, we would be looking for those regions with negative values. The roots would **not** be included, since they satisfy the *equality* and not the *inequality*.

BUT had the inequality included an equals sign (e.g. $x^2 + 2x \geq 3$), the roots themselves would be included in the solution set (indicated by closed circles), since the solution set would consist of regions with positive AND zero values for the left hand side.

Section 1.8: 48,56,66,84

- *Q48*: Solve, express in interval notation, and graph the solution set: $x^2 + 2x > 3$:

  Note that $x^2 + 2x > 3 \implies x^2 + 2x - 3 > 0$. So we want to identify the regions where the polynomial on the left-hand side of the inequality is positive.

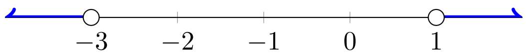
  Let us first find the roots.

$$\begin{aligned} x^2 + 2x - 3 &= 0 \\ (x+3)(x-1) &= 0 \\ x &= \{-3, 1\} \end{aligned}$$

  Now use the sign test to identify positive and negative regions.

  1. $x < -3$: Try $x = -4$: $(-4+3)(-4-1) = (-1)(-5) > 0$
  2. $-3 < x < 1$: Try $x = 0$: $(0+3)(0-1) = (3)(-1) < 0$
  3. $x > 1$: Try $x = 2$: $(2+3)(2-1) = (5)(1) > 0$

We want to only include positive regions in our solution set: $(-\infty - 3) \cup (1, \infty)$:



- *Q56*: Solve, express in interval notation, and graph the solution set: $4x^2(x^2 - 9) \leq 0$:

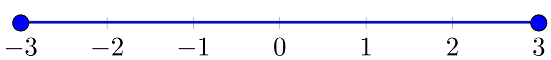
  Find roots:

$$4x^2(x+3)(x-3) = 0$$
$$x = \{-3, 0, 3\}$$

  Sign test. We want to include the roots, and then identify regions such that the left hand side of the inequality is less than 0. Note that $x^2 > 0$ no matter what, so we only need to check the sign of $(x+3)(x-3)$:

  1. $x \leq -3$: Try $x = -4$: $(-4+3)(-4-3) = (-1)(-7) > 0$
  2. $-3 \leq x \leq 0$: Try $x = -1$: $(-1+3)(-1-3) = (2)(-4) < 0$
  3. $0 \leq x \leq 3$: Try $x = 1$: $(1+3)(1-3) = (4)(-2) < 0$
  4. $x \geq 3$: Try $x = 4$: $(4+3)(4-3) = (7)(1) > 0$

  So the solution set is $[-3, 3]$:



- *Q66*: Solve, express in interval notation, and graph the solution set: $\frac{x}{x+1} > 3x$:

  First, bring everything to the left hand side:

$$\frac{x}{x+1} - 3x > 0$$
$$\frac{x - 3x(x+1)}{x+1} > 0$$
$$\frac{-3x^2 - 2x}{x+1} > 0$$

  To satisfy the inequality, the numerator and the denominator must have the same sign (this is what makes the left hand side greater than 0). That is, either

  1. $-3x^2 - 2x > 0, x + 1 > 0$
  2. $-3x^2 - 2x < 0, x + 1 < 0$

  For either of these to be true, we have the constraint that $x \neq -1$, since otherwise $x + 1 = 0$, which is not an allowable value based on the inequalities above. This must be left out of our solution set.

  From here we can identify the roots of the two equations derived from the inequalities above (that is, $-3x^2 - 2x = 0$ and $x + 1 = 0$) and then use the sign test on the single inequality:

$$\frac{-3x^2 - 2x}{x+1} > 0$$

  of course remembering that none of the roots are included in the solution set.

Aside: Note that if the inequality were instead

$$\frac{-3x^2 - 2x}{x+1} \geq 0$$

then the roots of the numerator would be included in the solution set, but the roots of the denominator would not. Why? Because in this case our constraints would be

1. $-3x^2 - 2x \geq 0, x + 1 > 0$
2. $-3x^2 - 2x \leq 0, x + 1 < 0$

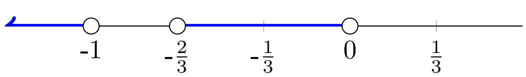
as the condition on the left hand side of the inequality to be greater than or equal to 0. In general, any value of x making the denominator 0 is not included in the solution set, while the inclusion of roots in the numerator depends on whether or not the inequality is strict.

$$\begin{aligned} -3x^2 - 2x &= 0 \\ \implies -x(3x+2) &= 0 \\ \implies x &= \{0, -\frac{2}{3}\} \end{aligned}$$

and we have $x = -1$ as our other root. Sign test:

1. $x < -1$: Try $x = -2$: $\frac{-3(-2)^2-2(-2)}{(-2)+1} = \frac{-12+4}{-1} = 8 > 0$
2. $-1 < x < -\frac{2}{3}$: Try $x = -0.75$: $\frac{-3(0.75)^2+2(0.75)}{-0.75+1} = \frac{-\frac{27}{16}+\frac{3}{2}}{\frac{1}{4}} = \frac{\frac{-27+24}{16}}{\frac{1}{4}} = -\frac{3}{16}\left(\frac{4}{1}\right) = -\frac{3}{4} < 0$
3. $-\frac{2}{3} < x < 0$: Try $x = 0.5$: $\frac{-3(0.5)^2+2(0.5)}{-0.5+1} = \frac{-\frac{3}{4}+1}{\frac{1}{2}} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2} > 0$
4. $x > 0$: Try $x = 1$: $\frac{-3(1)^2-2(1)}{(1)+1} = \frac{-5}{2} < 0$

So the solution set is $(-\infty, -1) \cup \left(-\frac{2}{3} < x < 0\right)$:

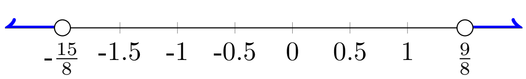


- *Q84*: Solve, express in interval notation, and graph the solution set: $|8x + 3| > 12$:

  Split into two inequalities ($8x + 3 > 12$ and $8x + 3 < -12$):

$$\begin{aligned} 8x + 3 > 12 &\implies x > \frac{9}{8} \\ 8x + 3 < -12 &\implies x < -\frac{15}{8} \end{aligned}$$

  The solution set, graphically:



Section 1.10: 50,72

- *Q50*: Find an equation of the line that satisfies: through (-2,-11); perpendicular to the line passing through (1,1) and (5,-1).

  We first need to find the slope of the line that is perpendicular to the line we want:

$$\begin{aligned} m_\perp &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-1-1}{5-1} = \frac{-2}{4} \\ &= -\frac{1}{2} \end{aligned}$$

  Now find the slope of the line we want, represented by the negative reciprocal of the slope we found:

$$\begin{aligned} m &= -(m_\perp)^{-1} \\ &= -\left(-\frac{1}{2}\right)^{-1} \\ &= -(-2) = 2 \end{aligned}$$

  So we want a line of slope two that goes through (-2,-11). Using the point slope formula:

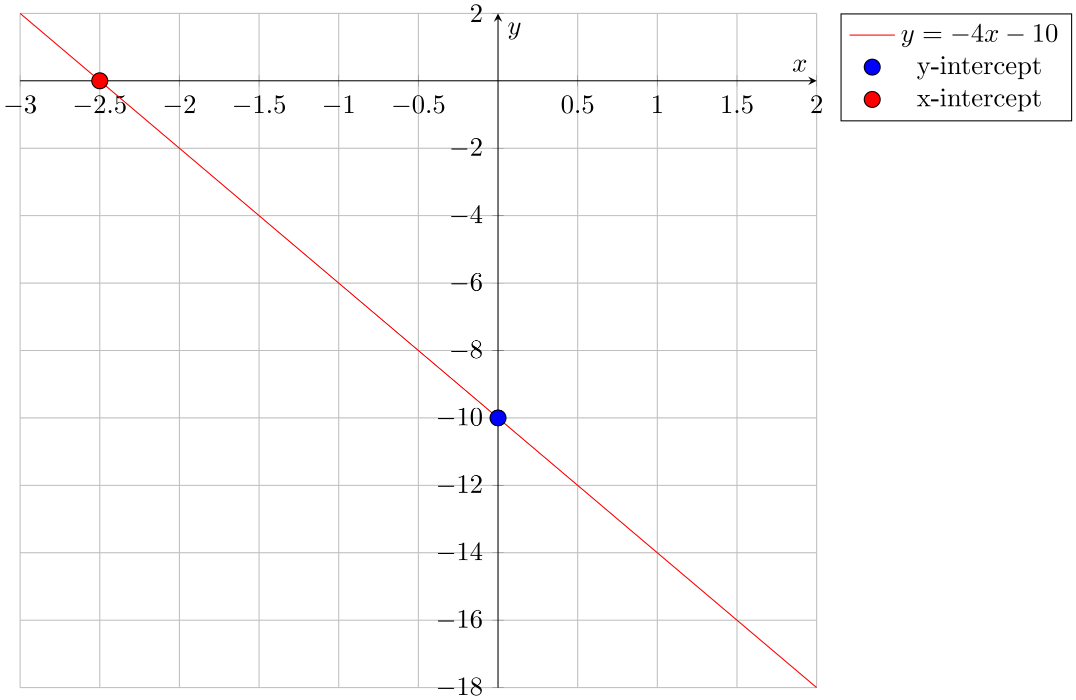
$$\begin{aligned} y - (-11) &= 2(x - (-2)) \\ y + 11 &= 2(x + 2) \\ y &= 2x + 4 - 11 \\ y &= 2x - 7 \end{aligned}$$

- *Q72*: Find the x- and y-intercepts of the line, and draw its graph: $y = -4x - 10$.

  For the x-intercept, solve the equation $-4x - 10 = 0$, which gives us $x = -\frac{5}{2} = -2.5$.

  For the y-intercept, substitute x=0: $y = -4(0) - 10 = -10$.

  The graph:



$y = -4x - 10$
y-intercept
x-intercept
x
y
-3
-2.5
-2
-1.5
-1
-0.5
0.5
1
1.5
2
2
-2
-4
-6
-8
-10
-12
-14
-16
-18